

Approximation Algorithms

Introduction

- Many problems of practical significance are NP-complete and can be solved in following ways:
 1. If the actual inputs are small, an algorithm with exponential running time may be perfectly satisfactory.
 2. Identify important special cases that can be solved in polynomial time.
 3. Approximation Algorithm
 - Algorithms that runs in polynomial time and always produce a solution close to the optimal.

Performance Ratios

- An algorithm for a problem has an approximation ratio of $\rho(n)$ if, for any input of size n , the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n)$$

- If an algorithm achieves an approximation ratio of $\rho(n)$, it is called as a $\rho(n)$ -approximation algorithm.
- $\rho(n) \geq 1$
 - 1-approximation algorithm produces an optimal solution.

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- Definitions of approximation ratio and $\rho(n)$ -approximation algorithm can be applied to both minimization and maximization problems.
- For a maximization problem,
 - $0 < C \leq C^*$, and the ratio C^*/C gives the factor by which the cost of an optimal solution is larger than the cost of the approximate solution.
- For a minimization problem,
 - $0 < C^* \leq C$, and the ratio C/C^* gives the factor by which the cost of the approximate solution is larger than the cost of an optimal solution.

Traveling Salesman Problem

Traveling Salesman Problem

- Given a complete undirected graph $G = (V, E)$ with a nonnegative integer cost $c(u, v)$ associated with each edge $(u, v) \in E$, find a hamiltonian cycle (a tour) of G with minimum cost.
- Consider two cases:
 - with and without triangle inequality.
 - c satisfies triangle inequality, if for all vertices $u, v, w \in V$,
$$c(u, w) \leq c(u, v) + c(v, w)$$
- Finding an optimal solution is NP-complete in both cases.

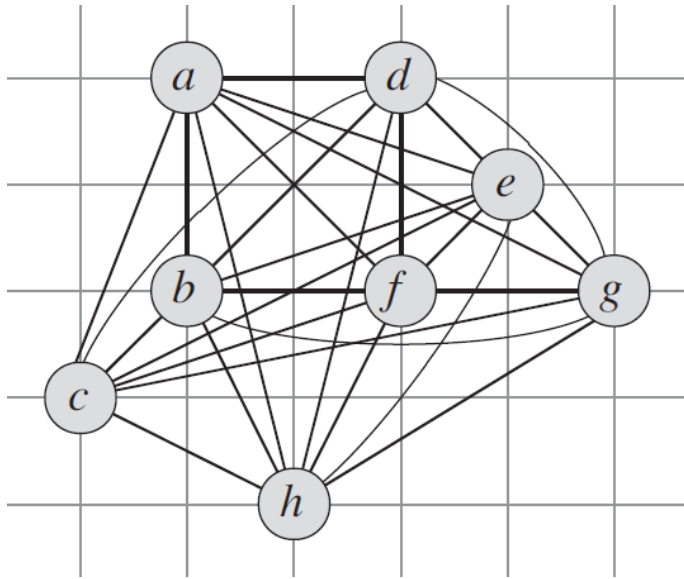
TSP with Triangle Inequality

- Compute a minimum spanning tree, whose weight gives a lower bound on the length of an optimal traveling-salesman tour.
- Use the minimum spanning tree to create a tour whose cost is no more than twice that of the minimum spanning tree's weight, as long as the cost function satisfies the triangle inequality.
- Assuming,
 - G – a complete undirected graph.
 - c – a cost function satisfying the triangle inequality.

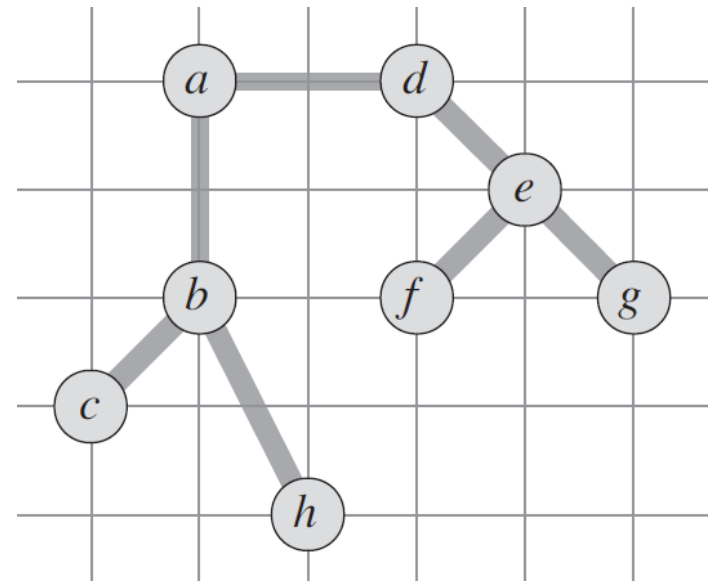
APPROX-TSP-TOUR(G, c)

1. Select a vertex $r \in G.V$ to be a “root” vertex
2. Compute a minimum spanning tree T for G from root r using MST-PRIM(G, c, r)
3. Let H be a list of vertices, ordered according to when they are first visited in a preorder tree walk of T
4. **return** the hamiltonian cycle H

Example

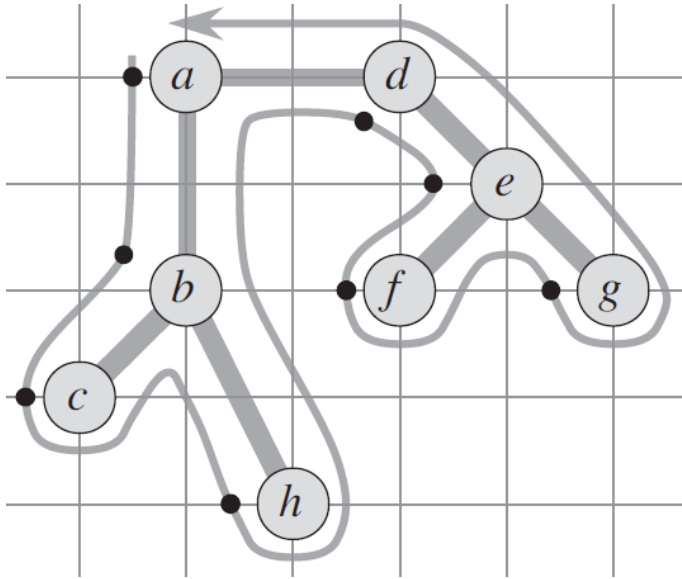


1. A complete undirected graph. Vertices lie on intersections of integer grid lines. For example, f is one unit to the right and two units up from h . The cost function between two points is the ordinary Euclidean distance.

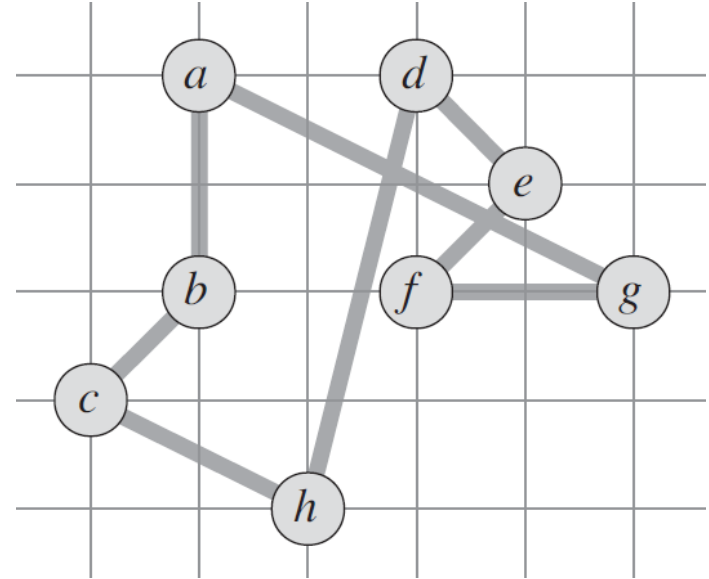


2. A minimum spanning tree T of the complete graph, as computed by MST-PRIM. Vertex a is the root vertex. Only edges in the minimum spanning tree are shown. The vertices happen to be labeled in such a way that they are added to the main tree by MST-PRIM in alphabetical order.

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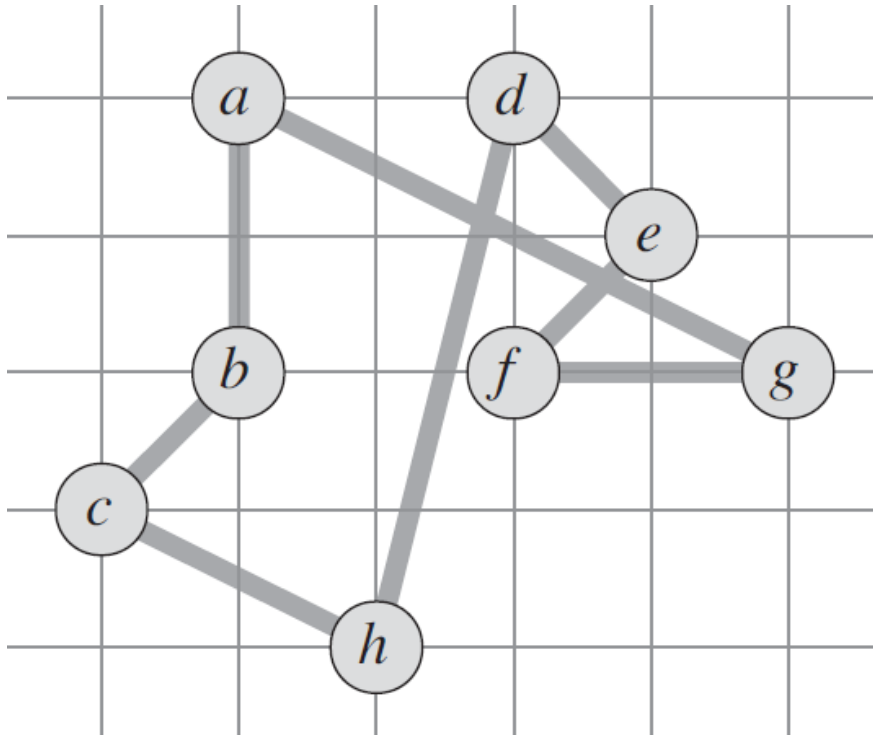
3. A walk of T , starting at a . A full walk of the tree visits the vertices in the order $a b c b h b a d e f e g e d a$. A preorder walk of T lists a vertex just when it is first encountered, as indicated by the dot next to each vertex, yielding the ordering $a b c h d e f g$.



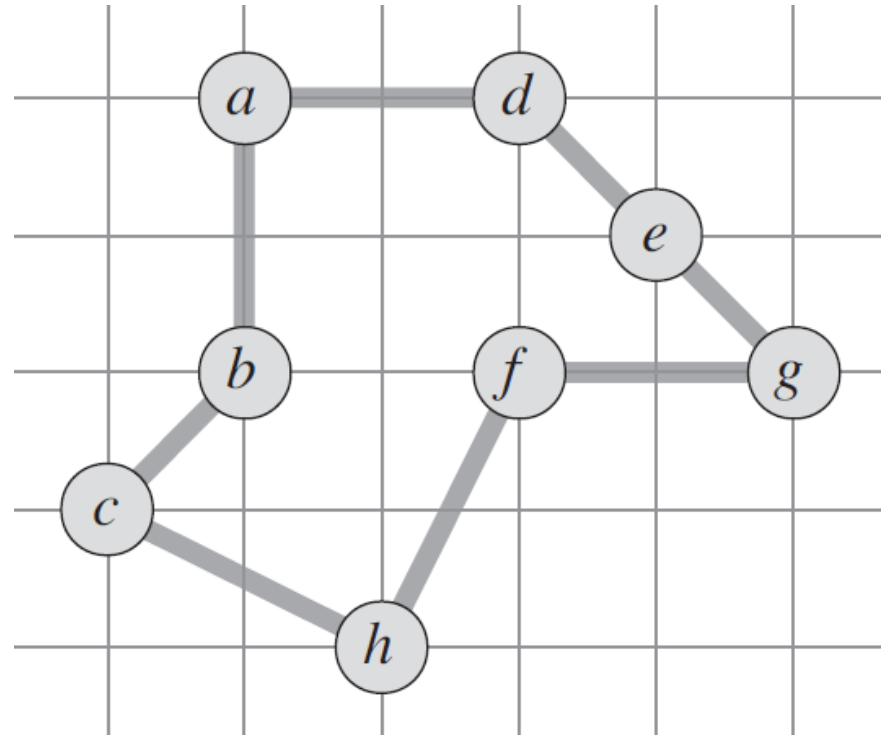
4. A tour obtained by visiting the vertices in the order given by the preorder walk, which is the tour H returned by APPROX-TSP-TOUR. Its total cost is approximately 19.074.

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It is known that APPROX-TSP-TOUR is a polynomial-time 2-approximation algorithm, i.e. $19.074 \leq 2 * 14.715$. The relation is clearly maintained for the considered example.



Tour H obtained using APPROX-TSP-TOUR.
Cost = 19.074

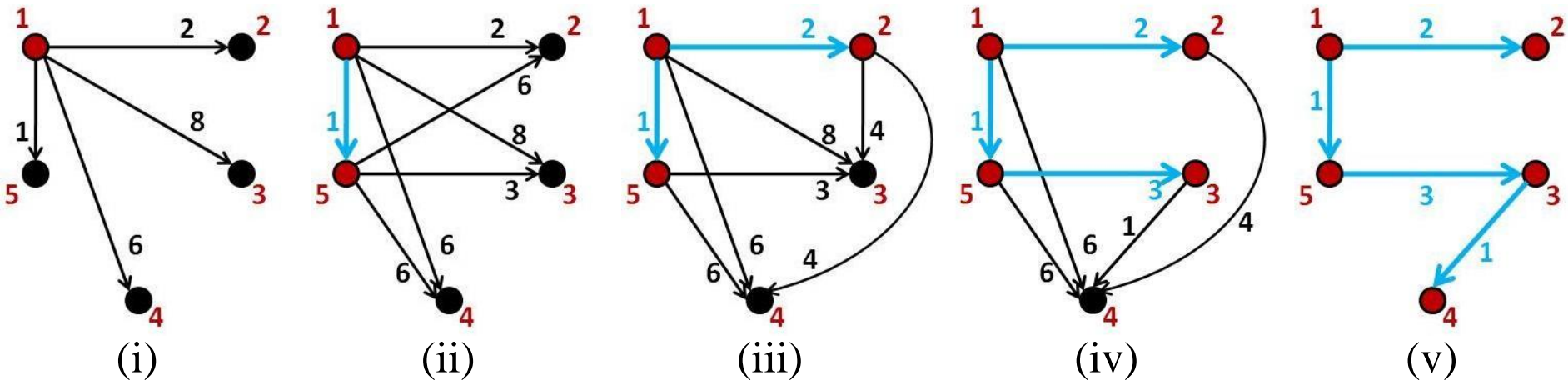


An optimal tour H* for the original complete graph. Cost = 14.715

Example

- Let the starting vertex be '1'.
- Computing MST using Prim's.

	1	2	3	4	5
1	0	2	8	6	1
2	1	0	4	4	2
3	5	3	0	1	5
4	4	7	2	0	1
5	2	6	3	6	0



- Preorder traversal of MST
 - 1 2 5 3 4, or
 - 1 5 3 4 2

Contd...

	1	2	3	4	5
1	0	2	8	6	1
2	1	0	4	4	2
3	5	3	0	1	5
4	4	7	2	0	1
5	2	6	3	6	0

- Computing tour cost.

– 1 2 5 3 4

$$\begin{aligned}\text{– Cost} &= c(1,2) + c(2,5) + c(5,3) + c(3,4) + c(4,1) \\ &= 2 + 2 + 3 + 1 + 4 = 12\end{aligned}$$

OR

– 1 5 3 4 2

$$\begin{aligned}\text{– Cost} &= c(1,5) + c(5,3) + c(3,4) + c(4,2) + c(2,1) \\ &= 1 + 3 + 1 + 7 + 1 = 13\end{aligned}$$

